

# $sl(2)$ VARIATIONAL SCHEMES FOR SOLVING ONE CLASS OF NONLINEAR QUANTUM MODELS

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## Abstract

Hamiltonians of a wide-spread class of  $G_{inv}$ -invariant nonlinear quantum models, including multiboson and frequency conversion ones, are expressed as non-linear functions of  $sl(2)$  generators. It enables us to use standard variational schemes, based on  $sl(2)$  generalized coherent states as trial functions, for solving both spectral and evolution tasks. In such a manner a new analytical expression is found for energy spectra in a mean-field approximation which is beyond quasi-equidistant ones obtained earlier.

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## 1 Introduction

For last decades a great attention has been paid to developments of both exact and approximate techniques to solve and examine different dynamical problems for quantum strongly coupled systems whose interaction Hamiltonians are expressed by nonlinear functions of operators describing subsystems (see, e.g., [1-9] and references therein). However, as a rule, such techniques either are adapted for treating special forms of model Hamiltonians and initial quantum states [1-5,7-9] or require lengthy and tedious calculations (as it is the case, e.g., for the algebraic Bethe ansatz [6]).

Recently, a new universal Lie-algebraic approach has been developed [10-13] to get exact solutions of both spectral and evolution problems for some nonlinear quantum models of strongly coupled subsystems having symmetry groups  $G_{inv}$ . It was based on exploiting a formalism of polynomial Lie algebras  $g_{pd}$  as dynamic symmetry algebras  $g^{DS}$  of models under study, and, besides, generators of these algebras  $g_{pd}$  can be interpreted as  $G_{inv}$ -invariant "essential" collective dynamic variables in whose terms model dynamics are described completely. Specifically, this approach enabled us to develop some efficient techniques for solving physical tasks in the case of  $g^{DS} = sl_{pd}(2)$ , when model Hamiltonians  $H$  are expressed as follows

$$H = aV_0 + gV_+ + g^*V_- + C, \quad [V_0, C] = 0, \quad V_- = (V_+)^+, \quad (1.1)$$

where  $C$  is a function of model integrals of motion  $R_i$  and  $V_0, V_{\pm}$  are the  $sl_{pd}(2)$  generators satisfying the commutation relations

$$[V_0, V_{\pm}] = \pm V_{\pm}, \quad [V_-, V_+] = \psi_n(V_0 + 1) - \psi_n(V_0),$$

$$\psi_n(V_0) = A \prod_{i=1}^n (V_0 + \lambda_i(\{R_j\})) \quad (1.2)$$

The structure polynomials  $\psi_n(V_0)$  depend additionally on  $\{R_i, i = 1, \dots\}$ , and their exact expressions for some wide-spread classes of concrete models were given in [10-12].

All techniques [10-13] are based on using expansions of most important physical quantities (evolution operators, generalized coherent states (GCS), eigenfunctions etc.) by power series in the  $sl_{pd}(2)$  shift generators  $V_{\pm}$  and on decompositions

$$L(H) = \sum_{\oplus} L([l_i]), \quad (V_+ V_- - \psi_n(V_0) \equiv -\psi_n(R_0)|_{L([l_i])} = 0 \quad (1.3)$$

of Hilbert spaces  $L(H)$  of quantum model states in direct sums of the subspaces  $L([l_i])$  which are irreducible with respect to joint actions of algebras  $sl_{pd}(2)$  and symmetry groups  $G_{inv}$  and describe specific " $sl_{pd}(2)$ -domains" evolving independently in time under action of the Hamiltonians (1.1);  $[l_0]$  are lowest weights of  $L([l_i])$ :  $\psi_n(l_0) = 0$  and other quantum numbers  $l_i, i = 1, \dots$  are eigennumbers of operators  $R_i$ . Then, using restrictions of Eqs. (1.1)-(1.2) on  $L([l_i])$ , one can develop simple algebraic calculation schemes for finding evolution operators

$$U_H(t) = \sum_{f=-\infty}^{\infty} V_+^f u(v_0; t), \quad V_+^{-f} \equiv V_-^f ([\psi_n(V_0)]^{(f)})^{-1}, \quad [\psi_n(x)]^{(f)} \equiv \prod_{r=0}^{f-1} \psi_n(x - r), \quad (1.4a)$$

amplitudes  $Q_v(E_f)$  of expansions

$$|E_f\rangle = A_f \prod_j (V_+ - \kappa_j^f) |[l_i]\rangle = \sum_v Q_v(E_f) |[l_i]; v\rangle \quad (1.4b)$$

of energy eigenstates  $|E_f\rangle$  in orthonormalized bases  $\{|[l_i]; v\rangle : V_0 |[l_i]; v\rangle = (l_0 + v) |[l_i]; v\rangle\}$  and appropriate energy spectra  $\{E_f\}$  of bound states [11,13]. (In fact, the factorized form of  $|E_f\rangle$  given by the first equality in (1.4b) realizes an efficient modification of the algebraic Bethe ansatz [6] in terms of collective dynamic variables related to the  $sl_{pd}(2)$  algebras [11,13].) In the paper [12] some explicit integral expressions were found for amplitudes  $Q_v(E)$ , eigenenergies  $\{E_a\}$  and "coefficients"  $u(v_0; t)$  of evolution operators  $U_H(t)$  with the help of a specific "dressing" (mapping) of solutions of some auxiliary exactly solvable tasks with the dynamic algebra  $sl(2)$ .

However, all exact results obtained do not yield simple working formulas for analysis of models (1.1) and revealing different physical effects (e.g., a structure of collapses and revivals of the Rabi oscillations [2,8], bifurcations of solutions [5] etc.) at arbitrary initial quantum states of models. Therefore, it is necessary to develop some simple techniques, in particular, to get some closed, perhaps, approximate expressions for evolution operators, energy eigenvalues and wave eigenfunctions, which would describe main important physical features of model dynamics with a good accuracy (cf. [5,8,9]). Below we examine some possibilities along these lines for models (1.1)-(1.2) by means of reformulating them in terms of the formalism of the usual  $sl(2)$  algebra and developing variational schemes corresponding to quasiclassical approximations for original models by analogy with developments [5,14-16].

## 2 A reduction of linear $sl_{pd}(2)$ problems to non-linear $sl(2)$ ones

We can reformulate models (1.1)-(1.2) in terms of  $sl(2)$  generators using an isomorphism of the  $sl_{pd}(2)$  algebras to extended enveloping algebras  $\mathcal{U}_{\psi}(sl(2))$  of the familiar algebra  $sl(2)$ .

This isomorphism is established via a generalized Holstein-Primakoff mapping given on each subspace  $L([l_i])$  as follows [10,11]

$$Y_0 = V_0 - l_0 \mp j, \quad Y_+ = V_+[\phi_{n-2}(Y_0)]^{-1/2}, \quad \phi_{n-2}(Y_0) = \frac{\psi_n(Y_0 + l_0 \pm j + 1)}{(j \mp Y_0)(\pm j + 1 + Y_0)}, \quad Y_- = (Y_+)^+,$$

$$[Y_0, Y_{\pm}] = \pm Y_{\pm}, \quad [Y_-, Y_+] = \mp 2Y_0 \quad (2.1)$$

where  $Y_{\alpha}$  are the  $sl(2)$  generators,  $\mp j$  are lowest weights of  $sl(2)$  irreducible representations realized on subspaces  $L([l_i])$  and  $\psi_2(x) = (j \pm x)(\pm j + 1 - x)$  are quadratic structure functions  $\psi_n(x) \equiv \psi_2(x)$  of  $sl(2)$  (hereafter upper/lower signs corresponding to the  $su(2)/su(1, 1)$  algebras are chosen for finite/infinite dimensions  $d([l_i])$  of the spaces  $L([l_i])$ ).

Note that, by definition, functions  $\phi_{n-2}(Y_0)$  on spaces  $L([l_i])$  can be chosen as polynomials of  $(n - 2)$ -th degree in  $Y_0$ . For example, substituting

$$\psi_3(V_0) = \frac{1}{4}(2V_0 + R_2 - R_1)(2V_0 + R_1 + R_2)(-V_0 + R_2 + 1),$$

$$l_0 = \frac{|k| - s}{3}, \quad l_1 = k, l_2 = \frac{|k| + 2s}{3}, \quad k = 0, \pm 1, \pm 2, \dots; \quad s = 2j = d([l_i]) - 1 = 0, 1, 2, \dots \quad (2.2)$$

for three-boson models [11-13]

$$H_{tb} = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \omega_3 a_3^+ a_3 + g(a_1^+ a_2^+) a_3 + g^*(a_1 a_2) a_3^+, \quad (2.3a)$$

$$V_0 = (N_1 + N_2 - N_3)/3, \quad V_+ = (a_1^+ a_2^+) a_3, \quad a = \omega_1 + \omega_2 - \omega_3, \quad N_i = a_i^+ a_i,$$

$$2C = R_1(\omega_1 - \omega_2) + R_2(\omega_1 + \omega_2 + 2\omega_3), \quad R_1 = N_1 - N_2, \quad 3R_2 = N_1 + N_2 + 2N_3 \quad (2.3b)$$

we get  $\phi_1(Y_0) = Y_0 + j + |k| + 1$ . Similar expressions can be found for  $\phi_1(Y_0)$  in the cases of the point-like Dicke and the second harmonic generation models taking appropriate expressions from [12].

Then, restrictions  $H_{[l_i]}$  of Hamiltonians (1.1) on  $L([l_i])$  may be re-written in terms of  $Y_{\alpha}$  as follows

$$H_{[l_i]} = aY_0 + Y_+ \tilde{g}(Y_0) + \tilde{g}^+(Y_0)Y_- + \tilde{C}, \quad \tilde{g}(Y_0) = g\sqrt{\phi_{n-2}(Y_0)}, \quad \tilde{C} = C + a(\pm j + l_0), \quad (2.4)$$

Evidently, this form corresponds to generalizations of semi-classical (linear in  $sl(2)$  generators) versions of matter-radiation interaction models [8,9,12] by introducing operator (intensity-dependent) coupling coefficients  $\tilde{g}(Y_0)$  (cf. [3,7]). Emphasize, however, a collective (not associated with a single subsystem) nature of operators  $Y_{\alpha}$  in Eq. (2.4) (cf. [9]); therefore, dynamic variables  $Y_{\alpha}$  correspond to a non-standard quasiclassical approximation (when  $\tilde{g}(Y_0) = \text{const}$  in Eq. (2.4)) of original models as it follows, e.g., from a direct comparison of such an approximation with standard (when creation/destruction operators of one mode are replaced by  $c$ -numbers) semiclassical limits for the model (2.3).

If  $n = 2$ , then  $\phi_{n-2}(Y_0) = 1$ ,  $sl_{pd}(2) = sl(2)$ ,  $l_0 = \pm j$ , and the formalism of GCS related to the  $SL(2)$  group displacement operators

$$S_Y(\xi = re^{i\theta}) = \exp(\xi Y_+ - \xi^* Y_-) = \exp[t(r)e^{i\theta} Y_+] \exp[-2 \ln c(r) Y_0] \exp[-t(r)e^{-i\theta} Y_-], \quad (2.5)$$

( $t(r) = \tan r / \tanh r, c(r) = \cos r / \cosh r$  for  $su(2)/su(1,1)$ ) yields a powerful tool for solving both spectral and evolution tasks [16].

Specifically, in this case, using the well-known  $sl(2)$  transformation properties of operators  $Y_\alpha$  under the action of  $S_Y(\xi)$  [16]:

$$S_Y(\xi)Y_+S_Y(\xi)^\dagger \equiv Y_+(\xi) = [c(r)]^2Y_+ \pm e^{-i\theta}[s(2r)Y_0 - e^{-i\theta}[s(r)]^2Y_-], \quad Y_-(\xi) = (Y_+(\xi))^\dagger,$$

$$S_Y(\xi)Y_0S_Y(\xi)^\dagger \equiv Y_0(\xi) = c(2r)Y_0 - \frac{s(2r)}{2}[e^{i\theta}Y_+ + e^{-i\theta}Y_-], \quad s(r) = \sin r / \sinh r, \quad (2.6)$$

Hamiltonians  $H_{[l_i]}$  can be transformed into the form

$$\tilde{H}_{[l_i]}(\xi) = S_Y(\xi)H_{[l_i]}S_Y(\xi)^\dagger = \tilde{C} + Y_0A_0(a, g; \xi) + Y_+A_+(a, g; \xi) + Y_-A_+^*(a, g; \xi) \quad (2.7a)$$

At the values  $\xi_0 = \frac{g}{2|g|}\arctan\frac{2|g|}{a}$  for  $su(2)$  and  $\xi_0 = \frac{g}{2|g|}\operatorname{arctanh}\frac{2|g|}{a}$  for  $su(1,1)$  of the parameter  $\xi$  one gets  $A_+(a, g; \xi) = 0$ , and the Hamiltonian  $\tilde{H}_{[l_i]}(\xi)$  takes the form

$$\tilde{H}_{[l_i]}(\xi_0) = \tilde{C} + Y_0\sqrt{a^2 \pm 4|g|^2} \quad (2.7b)$$

which is diagonal on eigenfunctions  $|[l_i]; v\rangle = \tilde{N}(j, v)(Y_+)^v|[l_i]; v=0\rangle$ ,  $N^{-2}(j, v) = v!(2j)!/(2j-v)!$  for  $su(2)$  and  $N^{-2}(j, v) = v!\Gamma(2j+v)/\Gamma(2j)$  for  $su(1,1)$ . Therefore, original Hamiltonians  $H_{[l_i]}$  have the eigenenergies

$$E_v([l_i]; \xi_0) = \tilde{C} + (\mp j + v)\sqrt{a^2 \pm 4|g|^2} \quad (2.8a)$$

and eigenfunctions

$$|[l_i]; v; \xi_0\rangle = S_Y(\xi_0)^\dagger|[l_i]; v; \rangle \quad (2.8b)$$

Similarly, when  $sl_{pd}(2) = sl(2)$ , operators  $S_Y(\xi(t))$  are "principal" parts in the evolution operators  $U_H(t) = \exp(i\phi(t)Y_0)S_Y(\xi(t))$  with  $c$ -number functions  $\phi(t), \xi(t)$  being determined from a set of non-linear differential equations corresponding to classical motions [16,17].

However, for arbitrary degrees  $n$  of  $\psi_n(V_0)$  Hamiltonians (2.4) are essentially non-linear in  $sl(2)$  generators  $Y_\alpha$ , and, therefore, the situation is very changed. Particularly, in general cases it is unlikely to diagonalize  $H_{[l_i]}$  with the help of operators  $S_Y(\xi)$  since analogs of Eq. (2.7a) on multi-dimensional spaces  $L([l_i])$

$$\tilde{H}_{[l_i]}(\xi) = S_Y(\xi)H_{[l_i]}S_Y(\xi)^\dagger = aY_0(\xi) + Y_+(\xi)\tilde{g}(Y_0(\xi)) + \tilde{g}^+(Y_0(\xi))Y_- + \tilde{C} \quad (2.9)$$

contain (after expanding them in power series) many terms with higher powers of  $Y_\pm$  [13].

Nevertheless, the formalism of the  $SL(2)$  group GCS  $|[l_i]; v; \xi\rangle = S_Y(\xi)^\dagger|[l_i]; v; \rangle$  [16] can be an efficient tool for analyzing non-linear models [5,11,14-16], in particular, for getting approximate analytical solutions. Specifically, a simplest example of such approximations was obtained in [11] by mapping (with the help of the change  $V_\alpha \rightarrow Y_\alpha$ ) Hamiltonians (1.1) by Hamiltonians  $H_{sl(2)}$  which are linear in  $sl(2)$  generators  $Y_\alpha$  (but with modified constants  $\tilde{a}, \tilde{g}$ ) and have on each fixed subspace  $L([l_i])$  equidistant energy spectra obtained from Eq. (2.8a). However, this (quasi)equidistant approximation, in fact, corresponding to a substitution of certain effective coupling constants  $\tilde{g}$  instead of true operator entities  $\tilde{g}(Y_0)$  in Eq. (2.4), does not enable to display many peculiarities of models (1.1) related to essentially non-equidistant parts

of their spectra. Therefore, it is needed in corrections, e.g., with the help of iterative schemes [8,14,15]; specifically, one may develop perturbative schemes by using expansions of operator entities  $\tilde{g}(Y_0)$  in Taylor series in  $Y_0$  as it was made implicitly for the Dicke model in [8,9]. But there exist a more effective, incorporating many peculiarities of models (1.1), way to amend the quasi-equidistant approximation.

### 3 $SL(2)$ energy functionals and variational schemes for solving spectral and evolution tasks

This way is in applying  $SL(2)$  GCS  $|[l_i]; v; \xi\rangle = S_Y(\xi)^\dagger |[l_i]; v; \rangle$  as trial functions in the variational schemes of determining energy spectra and quasiclassical dynamics [5,15]. Indeed, the results (2.8) are obtained by using a variational scheme determined by the stationarity conditions

$$a) \frac{\partial \mathcal{H}([l_i]; v; \xi)}{\partial \theta} = 0, \quad b) \frac{\partial \mathcal{H}([l_i]; v; \xi)}{\partial r} = 0 \quad (3.1)$$

for the energy functional  $\mathcal{H}([l_i]; v; \xi) = \langle [l_i]; v; \xi | H | [l_i]; v; \xi \rangle = \langle [l_i]; v | aY_0(\xi) + Y_+(\xi) + Y_- + \tilde{C} | [l_i]; v \rangle$ . At same time an appropriate quasiclassical dynamics, which is isomorphic to the exact quantum one when  $sl_{pd}(2) = sl(2)$  [14-16], is described by the classical Hamiltonian equations [5,14,16]

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}, \quad \mathcal{H} = \langle z(t) | H | z(t) \rangle \quad (3.2a)$$

for "motion" of the canonical parameters  $p, q$  of the  $SL(2)$  GCS  $|z(t)\rangle = \exp(-z(t)Y_+ + z(t)^*Y_-)|\psi_0\rangle$  as trial functions in the time-dependent Hartree-Fock variational scheme with the Lagrangian  $\mathcal{L} = \langle z(t) | (i\partial/\partial t - H) | z(t) \rangle$ ;  $p = j \cos \theta, q = \phi, z = \theta/2 \exp(-i\phi)$  for  $su(2)$  and  $p = j \cosh \theta, q = \phi, z = \theta/2 \exp(-i\phi)$  for  $su(1, 1)$ . An equivalent formulation in  $\mathbf{Y} = (Y_1, Y_2, Y_0)$  space can be given in terms of  $sl(2)$  Euler-Lagrange equations,

$$\dot{\mathbf{y}} = \frac{1}{2} \nabla \mathcal{H} \times \nabla C,$$

$$C = \pm y_0^2 + y_1^2 + y_2^2, \quad y_i = \langle z(t) | Y_i | z(t) \rangle, \quad y_{\pm} = y_1 \pm y_2, \quad \nabla = (\partial/\partial y_1, \partial/\partial y_2, \partial/\partial y_0) \quad (3.2b)$$

reducing to the well-known (linear) Bloch equations [5,14,17].

Similarly, general ideas of the analysis above and calculation schemes (3.1), (3.2) may be extended to the case of arbitrary polynomial algebras  $sl_{pd}(2)$  by using the energy functional  $\mathcal{H}([l_i]; v; \xi) = \langle [l_i]; v | \tilde{H}_{[l_i]}(\xi) | [l_i]; v \rangle$  with  $\tilde{H}_{[l_i]}(\xi)$  being given by Eq. (2.9). Naturally, results obtained in such a manner are not expected to coincide with exact solutions on all subspaces  $L([l_i])$  due to an essential nonlinearity of Hamiltonians (2.4) and their non-equivalence (unlike Eq. (2.7b)) to diagonal parts of Eq. (2.9); however, they yield, evidently, most close to exact "smooth" (analytical) solutions (cf. [5,14]). Without dwelling on a discussion of all aspects of such an extension we consider in detail an application of the procedure (3.1) to the most wide-spread class [11] of Hamiltonians (2.4) with the  $su(2)$  dynamic symmetry which includes the model (2.3).

Note that the condition (3.1a) gives  $e^{i\theta} = g/|g|$  as in the linear case, and, due to the form of trial functions, it is sufficiently to solve Eq. (3.1b) only for finding ground states  $|[l_i]; v = 0; \xi\rangle$ .

Then, expanding r.h.s. of Eq. (2.5) in  $Y_\alpha$  power series and taking into account defining relations for the  $su(2)$  algebra one gets after some algebra the following expressions

$$\begin{aligned}
E_v^{su(2)}([l_i]; \xi_0) &= \mathcal{H}([l_i]; v; \xi_0) = C + a(l_0 + j) + a(-j + v) \cos 2r - 2|g| \sum_{f \geq 0} E_f^\phi(r; j; v), \\
E_f^\phi(r; j; v) &= E_f^\phi(r; j; 0) \left( \frac{1}{2} \sin 2r \right)^{-2v} \frac{(f)!(2j - v)!(f + 1)!}{(2j)!v!(f - v)!(f + 1 - v)!} \times \\
&\quad F(-v, -v + 2j + 1; f - v + 1; \sin^2 r) F(-v, -v + 2j + 1; f - v + 2; \sin^2 r), \\
E_f^\phi(r; j; 0) &= (\cos^{4j} r) \frac{(\tan r)^{2f+1} (2j)!}{(f)!(2j - f - 1)!} \sqrt{\phi_{n-2}(-j + f)}, \quad \phi_{n-2}(-j + f) = \frac{\psi(l_0 + 1 + f)}{(2j - f)(f + 1)},
\end{aligned} \tag{3.3}$$

with  $F(\dots)$  being the Gauss hypergeometric function [18], for energy eigenvalues  $E_v^{su(2)}([l_i]; \xi_0 = rg/|g|)$  where diagonalizing values of the parameter  $r$  are determined from solving the algebraic equation

$$0 = \sum_{f \geq 0} \frac{\alpha^{2f}}{(2j - 1 - f)!f!} \left\{ \frac{a\alpha}{|g|} - [4\alpha^2 j - (1 + \alpha^2)(2f + 1)] \sqrt{\phi_{n-2}(-j + f)} \right\}, \quad \alpha = -\tan r \tag{3.4}$$

For the case of the  $su(1, 1)$  dynamic symmetry Eqs. (3.3), (3.4), retaining their general structure form, are slightly modified due to differences in the definition (2.5) of  $S_Y(\xi)$  for  $su(2)$  and  $su(1, 1)$ . Let us make some remarks concerning this result.

1) As is seen from Eq. (3.3), its general structure coincides with the energy formula given by the algebraic Bethe ansatz [6], and spectral functions  $E_f^\phi(r; j; v)$  are non-linear in the discrete variable  $v$  labeling energy levels that provides a non-equidistant character of energy spectra within fixed subspaces  $L([l_i])$  at  $d([l_i]) > 3$ . Besides, due to the square roots in expressions for these functions different eigenfrequencies  $\omega_v^{su(2)} \equiv E_v^{su(2)}/\hbar$  are incommensurable:  $m\omega_{v_1}^{su(2)} \neq n\omega_{v_2}^{su(2)}$  that is an indicator of an origin of collapses and revivals of the Rabi oscillations [2,8] as well as of pre-chaotic dynamics [19]. Note that this dependence is impossible to get by using GCS related to uncoupled subsystems.

2) The r.h.s. of Eq. (3.4) is a polynomial of the degree  $2j + 1 = d([l_i])$ , and, in general, Eq. (3.4) may have  $2j + 1$  different roots  $r_i$  corresponding to  $2j + 1$  different stationary values of the energy functional  $\mathcal{H}([l_i]; v; \xi)$ . Therefore, one may assume that it is possible to get more simple expressions for  $E_f^\phi(r; j; v)$  with any  $v$  using  $E_f^\phi(r; j; 0)$  with different roots  $r_i$ . Note that this conjecture is valid for little dimensions  $d([l_i])$  when Eqs. (3.3)-(3.4) give exact results. Another way to modify and to simplify the results above is in using different properties, including integral representations, of the hypergeometric functions  $F(a, b; c; x)$ ; specifically, using relations between hypergeometric functions [18], one can express spectral functions  $E_f^\phi(r; j; v)$  in terms of the hypergeometric functions  ${}_4F_3(\dots; 1)$  (which are proportional to the  $sl(2)$  Racah coefficients).

3) Evidently, Eq. (3.3) generalizes Eq. (2.8a) for the (quasi)equidistant approximation abovementioned. Indeed, when replacing functions  $\phi_{n-2}(-j + f)$  by their certain "average" values, series in (3.3), (3.4) are summed up, and Eq. (3.3) is reduced to Eq. (2.8a); Taylor series expansions of functions  $\sqrt{\phi_{n-2}(-j + f)}$  provide perturbative corrections related to higher

degrees of the an-harmonicity of Hamiltonians (2.4). Furthermore, we can get an intermediate approximation for energy spectra if replacing in Eqs. (3.1) the exact energy functionals  $\mathcal{H}([l_i]; v; \xi) = \langle [l_i]; v | \tilde{H}_{[l_i]}(\xi) | [l_i]; v \rangle$  by their mean-field (corresponding to the Ehrenfest theorem) approximations

$$\begin{aligned} \mathcal{H}^{mfa}([l_i]; v; \xi) &= a \langle Y_0(\xi) \rangle + \langle Y_+(\xi) \rangle \tilde{g}(\langle Y_0(\xi) \rangle) + \tilde{g}^+(\langle Y_0(\xi) \rangle) \langle Y_-(\xi) \rangle + \tilde{C}, \\ \langle Y_\alpha(\xi) \rangle &= \langle [l_i]; v | Y_\alpha(\xi) | [l_i]; v \rangle \end{aligned} \quad (3.5)$$

Then Eqs. (3.3)-(3.4) are very simplified retaining their main characteristic features. For example, for the model (2.3) we find

$$E_v^{mfa}([l_i]; \xi_0) =$$

$$C + a(l_0 + j) + a(-j + v) \cos 2r - 2|g|(j - v) \sin 2r \sqrt{(-j + v) \cos 2r + j + |k| + 1} \quad (3.6a)$$

where  $r$  is determined from the equation

$$\frac{a}{2|g|} \sin 2r = \cos 2r \sqrt{2j \sin^2 r + |k| + 1} + \frac{j \sin^2 2r}{2\sqrt{2j \sin^2 r + |k| + 1}} \quad (3.6b)$$

(Similar expressions can be found for the point-like Dicke and the second harmonic generation models.) Besides, substituting Eq. (3.5) in Eqs. (3.2) one may get a mean-field approximation for dynamics equations reducing in the  $\mathbf{Y}$  space representation to non-linear Bloch equations (cf.[5,11]) obtained from Eqs. (3.2b) by the substitution

$$\begin{aligned} \nabla \mathcal{H} &= ([g + g^*][\phi_1(y_0)]^{1/2}, [g - g^*][\phi_1(y_0)]^{1/2}, a + \frac{1}{2}[g(y_1 + y_2) + g^*(y_1 - y_2)][\phi_1(y_0)]^{-1/2}), \\ \nabla C &= 2(y_1, y_2, y_0), \quad \phi_1(y_0) = y_0 + j + |k| + 1 \end{aligned} \quad (3.7)$$

4) Finally, Eqs. (3.3) and (3.6a) can be used for obtaining appropriate approximations

$$U_H^{su(2)/mfa}(t) = \sum_{[l_i], v} S_Y(\xi_0)^\dagger \exp\left(\frac{-it\omega_v^{su(2)/mfa}}{\hbar}\right) |[l_i]; v\rangle \langle [l_i]; v| S_Y(\xi_0) \quad (3.8)$$

for the evolution operators which are transformed to the form (1.4a) with the help of the standard group-theoretical technique [20].

## 4 Conclusion

So, we have obtained new approximations for energy spectra and evolution equations of models (1.1) by means of using the mapping (2.1) and the variational schemes (3.1), (3.2) with the  $SL(2)$  GCS as trial functions. They may be called as a "smooth"  $sl(2)$  quasiclassical approximations since they, in fact, correspond to picking out "smooth" (analytical)  $sl(2)$  factors  $\exp(\xi_0 Y_+ - \xi_0^* Y_-)$  in exact diagonalizing operators  $S(\tilde{\xi})$  and in the evolution operator  $U_H(t)$ . These approximations may be used for calculations of evolution of different quantum statistical

quantities (cf. [8,14]) and for determining bifurcation sets of non-linear Hamiltonian flows in parameter space (cf. [5]).

Further investigations may be related to a search of suitable multi-parametric specifications of exact diagonalizing operators  $S(\hat{\xi}) = S([\xi_0, \xi_1, \xi_2, \dots])$  using  $\exp(\xi_0 Y_+ - \xi_0^* Y_-)$  as initial ones in iterative schemes which are similar to those developed to examine non-linear problems of classical mechanics and optics [21] or as "principal" factors in the diagonalization schemes like (2.7) for Hamiltonians (1.1). From the practical point of view an important question is to get estimations of accuracy of approximations obtained and to make comparisons of their efficiency with other approximations (e.g., given in [8,9,11]). For the model (2.3) (and other ones with the structure polynomial  $\psi_3(x)$  of the third degree) it is of interest to compare results of approximations found above with exact calculations obtained by considering solvable cases of models under study. One of latters is given by integral solutions [12] and other may be yielded by the Riccati equations arising from a differential realization of  $sl_{pd}(2)$  generators  $V_\alpha$  [13]:

$$V_- = d/dz, \quad V_0 = zd/dz + l_0, \quad V_+ = \psi_n(zd/dz + l_0)(d/dz)^{-1} \quad (4.1)$$

which is, in turn, related to a realization of  $sl_{pd}(2)$  generators  $V_\alpha$  by quadratic forms in  $sl(2)$  generators  $Y_\alpha$  (cf. [15,22]). (In fact, this realization was used implicitly for obtaining exact integral solutions [12].) Besides, it is also of interest to investigate possible connections of these results with quasi-exactly solvable  $sl(2)$  models [23,24].

The work along these lines is now in progress.

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